

# Quantum Holonomy in Three-dimensional General Covariant Field Theory and Link Invariant

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## Abstract

We consider quantum holonomy of some three-dimensional general covariant non-Abelian field theory in Landau gauge and confirm a previous result partially proven. We show that quantum holonomy retains metric independence after explicit gauge fixing and hence possesses the topological property of a link invariant. We examine the generalized quantum holonomy defined on a multi-component link and discuss its relation to a polynomial for the link.

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Some time ago it was shown that quantum holonomy in a three-dimensional general covariant non-Abelian gauge field theory possesses topological information of the link on which the holonomy operator is defined [1]. The quantum holonomy operator was shown to be a central element of the gauge group so that, in a given representation of the gauge group,

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it is a matrix that commutes with the matrix representations of all other operators in the group. In an irreducible representation, it is proportional to the identity matrix. Quantum holonomy should therefore in general have more information on the link invariant than the quantum Wilson loop which, for the  $SU(2)$  Chern-Simons quantum field theory, was shown by Witten [2] to yield the Jones polynomial [3]. Horne [4] extended Witten's result to some other Lie groups. The difference between quantum holonomy and the Wilson loop becomes apparent in the  $SU(N|N)$  Chern-Simons theory, where the quantum Wilson loop vanishes identically for any link owing to the property of super-trace, but the quantum holonomy [1] yields the important Alexander-Conway polynomial [5–8].

However, the argument used in Ref. [1] was based only on the formal properties of the functional integral and complications that may arise from the necessity for gauge fixing in any actual computation were not taken into consideration. In addition, in a case when a metric is needed for gauge fixing, the metric independence of quantum holonomy may be violated. Furthermore, in the standard Faddeev-Popov technique used for gauge fixing, ghost fields and auxiliary fields that are introduced reduce the original local gauge symmetry to BRST symmetry, and it is no longer certain the formal arguments and manipulations used in Ref. [1] to derive its results are still valid. As well, the case of the quantum holonomy defined on multi-component links was not explicitly considered.

The main aim of the present work is to clarify these problems. We explicitly work in the Landau gauge<sup>1</sup> and confirm the results obtained in Ref. [1] for the case of a one-component contour. We then show that a quantum holonomy operator defined on a  $n$ -component link, which by construction is a tensor product of those operators defined on the components, is a central element of the universal enveloping algebra of the Lie algebra of the gauge group and, when evaluated on a set of  $n$  irreducible representations of the gauge group, has a

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<sup>1</sup> The reason why we prefer this gauge is because it allows one to avoid the infrared divergence in low dimensional gauge theories; for a discussion on Chern-Simons theory, see [15].

uniquely defined eigenvalue that is a polynomial invariant of the link.

The quantum holonomy is defined as

$$Z[C] \equiv \frac{1}{V} \int \mathcal{D}A \exp(iS[A]) f[A, C], \quad f[A, C] \equiv P \exp \left( i \oint_C A \right), \quad (1)$$

where where  $C$  is a contour in the three-dimensional manifold  $M$ ,  $S[A]$  is the action of some three-dimensional general covariant non-Abelian gauge field theory,  $P$  means path-ordering and  $V = \int \mathcal{D}g$  is gauge invariant group volume. Now we choose the Lorentz gauge condition

$$F[A] = \partial_\mu (\sqrt{-G} G^{\mu\nu} A_\nu) = 0, \quad G = \det(G_{\mu\nu}), \quad (2)$$

where  $G_{\mu\nu}$  is the metric of space-time manifold. According to standard Faddeev-Popov procedure, we insert the identity

$$1 \equiv \Delta_F[A] \int \mathcal{D}g \Pi_x \delta(F[A^g(x)]) \quad (3)$$

into Eq.(1) and obtain

$$Z[C] = \frac{1}{V} \int \mathcal{D}A \Delta_F[A] \int \mathcal{D}g \Pi_x \delta(F[A^g(x)]) \exp(iS[A]) f[A, C]. \quad (4)$$

Denoting  $A^g$  as  $A$  and replacing the original  $A$  by  $A^{g^{-1}}$ , we rewrite Eq.(4) as follows

$$\begin{aligned} Z[C] &= \frac{1}{V} \int \mathcal{D}g \mathcal{D}A^{g^{-1}} \Delta_F[A^{g^{-1}}] \Pi_x \delta(F[A]) \exp(iS[A^{g^{-1}}]) f[A^{g^{-1}}, C] \\ &= \frac{1}{V} \int \mathcal{D}g \mathcal{D}A \Delta_F[A] \Pi_x \delta(F[A]) \exp(iS[A]) f[A^{g^{-1}}, C] \\ &= \frac{1}{V} \int \mathcal{D}g \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \left\{ \exp \left[ iS[A] - i \int d^3x \sqrt{-G} G^{\mu\nu} (\partial_\mu B^a A_\nu^a - \partial_\mu \bar{c}^a D_\nu c^a) \right] \right. \\ &\quad \left. \times f[A^{g^{-1}}, C] \right\} \end{aligned} \quad (5)$$

where we have used the gauge invariance of  $S[A]$  and  $B^a(x), \bar{c}^a(x), c^a(x)$  are respectively auxiliary fields, ghost and antighost fields.

We perform the following maneuver on Eq.(5). Suppose  $g'$  is a global group element, write  $A$  as  $A^{g'g'^{-1}}$  and rename  $A^{g'^{-1}}$  as  $A$  and hence the original  $A$  is replaced by  $A^{g'}$ . We thus obtain,

$$\begin{aligned}
Z[C] &= \frac{1}{V} \int \mathcal{D}g \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \left\{ \exp \left[ iS[A] + i \int d^3x \sqrt{-G} (B^a \partial^\mu A_\mu^a - \bar{c}^a \partial_\mu D^\mu c^a) \right] \right. \\
&\quad \left. \times f[(A^{g^{-1}})^{g'}, C] \right\} \\
&= \frac{1}{V} \int \mathcal{D}g \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \left\{ \exp \left[ iS[A] + i \int d^3x \sqrt{-G} (B^a \partial^\mu A_\mu^a - \bar{c}^a \partial_\mu D^\mu c^a) \right] \right. \\
&\quad \left. \times \Omega_{g'}^{-1} f[A^{g^{-1}}, C] \Omega_{g'} \right\} \\
&= \Omega_{g'}^{-1} Z[C] \Omega_{g'} , \tag{6}
\end{aligned}$$

where we have used the fact that all the fields are in the adjoint representation of gauge group. Since Eq.(6) is true for every global gauge transformations, according to Schur's lemma, we conclude that when  $Z[C]$  is valued in an irreducible representation  $\rho$  it has the form,

$$\rho(Z[C]) = F[C] \mathbf{1}_\rho , \tag{7}$$

where  $\mathbf{1}_\rho$  is the matrix representation of the identity element in  $\rho$  and  $F[C]$ , the eigenvalue of  $Z[C]$  in  $\rho$ , is a scalar function depending on the contour  $C$ . Eq.(7) is one of the results obtained in Ref. [1] without explicit gauge fixing.

In the following, we shall show explicitly the metric independence of quantum holonomy. All fields are valued in the adjoint representation of gauge group. As a first step, Eq.(5) can be rewritten as follows

$$Z[C] = \frac{1}{V} \int \mathcal{D}g \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \exp \left\{ iS[A] + i \int d^3x \sqrt{-G} G^{\mu\nu} \delta_B [\bar{c}^a \partial_\mu A_\nu^a] \right\} f[A^{g^{-1}}, C] , \tag{8}$$

where the BRST transformations are:

$$\delta_B A^a = D_\mu c^a , \quad \delta_B c^a = \frac{1}{2} f^{abc} c^b c^c , \quad \delta_B \bar{c}^a = B^a , \quad \delta_B B^a = 0 . \tag{9}$$

Assuming that the functional measures of fields have no dependence on the metric, we have that

$$-\frac{2i}{\sqrt{-G}} \frac{\delta Z[C]}{\delta G^{\mu\nu}} = -\frac{2i}{\sqrt{-G}} \frac{\delta F[C]}{\delta G^{\mu\nu}} \mathbf{1}_\rho$$

$$\begin{aligned}
&= -\frac{2i}{\sqrt{-G}} \frac{\delta}{\delta G^{\mu\nu}} \left\{ \frac{1}{V} \int \mathcal{D}g \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \exp i \left[ iS[A] + i \int d^3x \sqrt{-G} G^{\alpha\beta} \delta_B (\bar{c}^a \partial_\alpha A_\beta^a) \right] \right. \\
&\quad \left. \times f[A^{g^{-1}}, C] \right\} \\
&= \frac{1}{V} \int \mathcal{D}g \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \exp \left[ iS[A] + i \int d^3x \sqrt{-G} G^{\alpha\beta} \delta_B (\bar{c}^a \partial_\alpha A_\beta^a) \right] T_{\mu\nu} f[A^{g^{-1}}, C], \quad (10)
\end{aligned}$$

where  $T_{\mu\nu}$  is the canonical symmetric energy-momentum,

$$\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-G}} \frac{\delta S_{eff}}{\delta G^{\mu\nu}} \\
&= A_\mu^a \partial_\nu B^a + A_\nu^a \partial_\mu B^a - \partial_\mu \bar{c}^a (D_\nu c)^a - \partial_\nu \bar{c}^a (D_\mu c)^a \\
&\quad - G_{\mu\nu} [A_\alpha^a \partial^\alpha B^a - \partial_\alpha \bar{c}^a (D^\alpha c)^a], \\
S_{eff} &= S[A] + \int d^3x \sqrt{-G} G^{\alpha\beta} \delta_B (\bar{c}^a \partial_\alpha A_\beta^a). \quad (11)
\end{aligned}$$

It can be written as a BRST trivial form from a careful observation,

$$\begin{aligned}
T_{\mu\nu} &= \delta_B \Theta_{\mu\nu}, \\
\Theta_{\mu\nu} &= -\partial_\mu \bar{c}^a A_\nu^a - \partial_\nu \bar{c}^a A_\mu^a - G_{\mu\nu} \partial^\alpha \bar{c}^a A_\alpha^a. \quad (12)
\end{aligned}$$

So we can obtain that

$$\begin{aligned}
&-\frac{2i}{\sqrt{-G}} \frac{\delta Z[C]}{\delta G^{\mu\nu}} = -\frac{2i}{\sqrt{-G}} \frac{\delta F(C)}{\delta G^{\mu\nu}} \mathbf{1}_\rho \\
&= \int \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \exp \left\{ iS[A] + i \int d^3x \sqrt{-G} G^{\alpha\beta} \delta_B (\bar{c}^a \partial_\alpha A_\beta^a) \right\} \delta_B \Theta_{\mu\nu} \frac{1}{V} \int \mathcal{D}g f[A^{g^{-1}}, C] \\
&= \langle 0 | [\hat{Q}_B, \hat{\Theta}_{\mu\nu}] \frac{1}{V} \int \mathcal{D}g f[\hat{A}^{g^{-1}}, C] | 0 \rangle = \langle 0 | \left[ \hat{Q}_B, \hat{\Theta}_{\mu\nu} \frac{1}{V} \int \mathcal{D}g f[\hat{A}^{g^{-1}}, C] \right] | 0 \rangle \\
&= 0, \quad (13)
\end{aligned}$$

where  $Q_B$  is the BRST charge corresponding to the BRST transformation Eq.(9) and the symbol hat “ $\hat{\cdot}$ ” denotes an operator. In the above, we have used the physical state condition in BRST quantization:  $\hat{Q}_B |\text{phys}\rangle = 0$  [9] and the fact that  $\frac{1}{V} \int \mathcal{D}g f[A^{g^{-1}}, C]$  is gauge invariant,

$$\left[ \hat{Q}_B, \frac{1}{V} \int \mathcal{D}g f[\hat{A}^{g^{-1}}, C] \right] = 0 \quad (14)$$

as well as the explicit corollary observed by Witten [10]: for two operators  $\hat{A}$  and  $\hat{B}$ , if  $[\hat{Q}_B, \hat{A}] = 0$ , then  $\hat{A}[\hat{Q}_B, \hat{B}] = [\hat{Q}_B, \hat{A}\hat{B}]$ . Therefore from Eq.(13),  $Z[C]$  and  $F[C]$  are generally covariant.

We shall now justify the assumption that there is no metric independence for the path integral measure. This assumed property was very crucial in the proof of the general covariance of quantum holonomy. The justification can be made from two aspects. First, a metric dependence in the path integral measure means that under metric variation the path integral measure has a nontrivial Jacobian factor. According to Fujikawa [11], this implies that the theory would have a conformal anomaly, i.e., the trace of the energy-momentum  $\langle \Theta_\mu^\mu \rangle$  does not vanish. Since the trace of the energy-momentum is proportional to the  $\beta$ -function of the theory [12], the existence of conformal anomaly would mean that the theory is not finite. On the other hand, it has been proved that three-dimensional topological field theories such as Chern-Simons [13] and  $BF$  [14] theories are finite to any order, that is, the  $\beta$ -function and anomalous dimension vanish identically. This has also been verified by explicit computation in concrete regularization schemes to two-loop [15–17]. Therefore the conformal anomaly and hence the metric dependence of the path integral measure should not exist.

We could also approach the issue from the opposite direction and impose the conformal anomaly-free condition. Then the Jacobian associated with the metric variation would be trivial, which means that the correct path integration variables would not be the original fields  $\Phi = (A, B, c, \bar{c})$  but would be the appropriate tensorial densities  $\tilde{\Phi}$  given by

$$\tilde{\Phi}_i = (-G)^{-w_i/2} \Phi_i, \quad (15)$$

where  $G$  is the determinant of metric tensor  $G_{\mu\nu}$ , the subscript “ $i$ ” labels a specific field in  $\Phi$  and  $w_i$  is the weight associated with the field  $\Phi_i$  whose value depends on the tensorial character of the corresponding field. Replacing  $\Phi_i$  by  $\tilde{\Phi}_i$  as the path integral variables and at the same time rewriting the action in terms of these new variables, we could make the path integral measure metric independent. This procedure would transfer the metric dependence of path integral measure to the effective action, and this would affect all the symmetries

of the effective action such as BRST symmetry etc. In Ref. [18], it was shown that for a cohomological topological field theory, whose action can always be written as a BRST-trivial form, this line of reasoning can be used to define an invariant path integral measure so that the topological character of the theory is preserved. However, for topological field theories of the Chern-Simons type, whose action cannot be written as a BRST commutator, it is not clear how this technique can be applied. We intend to explore this problem in detail elsewhere.

Now we try to understand the result Eq.(7) from the operator viewpoint. Since the global gauge symmetry is not affected by gauge fixing, the Noether current and charge corresponding to global gauge transformation are respectively,

$$j_\mu^a = -i \text{Tr} \frac{\partial \mathcal{L}}{\partial \partial^\mu \Phi} [T^a, \Phi], \quad Q^a = \int d^3x j_0^a, \quad (16)$$

where  $\Phi \equiv (A, B, c, \bar{c})$ . After quantization, since there is no anomaly in three dimensional gauge theory, we have

$$\partial^\mu \langle \hat{j}_\mu^a \rangle = 0, \quad \hat{Q}^a |0\rangle = 0. \quad (17)$$

The  $\hat{Q}^a$  constitute the operator realization of the generators of the global gauge group,

$$[\hat{Q}^a, \hat{Q}^b] = if^{abc} \hat{Q}^c. \quad (18)$$

Correspondingly,  $U_{g'} = \exp[-i\xi^a \hat{Q}^a]$  are global gauge group elements,  $\xi^a$  are group parameters. Under a global gauge transformation, the holonomy operator  $f[\hat{A}, C]$  transforms as

$$f[\hat{A}, C] \longrightarrow f[\hat{A}, C]^{g'} = U_{g'}^{-1} f[\hat{A}, C] U_{g'} = \Omega_{g'}^{-1} f[A, C] \Omega_{g'}, \quad (19)$$

where  $\Omega_{g'} = \exp[-i\xi^a T^a]$  are the matrix representations of group elements. So we have

$$\begin{aligned} Z[C] &= \langle 0 | f[\hat{A}, C] | 0 \rangle = \langle 0 | U_{g'}^{-1} f[\hat{A}, C] U_{g'} | 0 \rangle \\ &= \langle 0 | \Omega_{g'}^{-1} f[A, C] \Omega_{g'} | 0 \rangle = \Omega_{g'}^{-1} Z[C] \Omega_{g'}. \end{aligned} \quad (20)$$

$Z[C]$  commutes with every global gauge transformation, and from Schur's lemma we obtain Eq.(7).

Finally let us consider the generalized quantum holonomy defined on a multi-component link  $L$ , the disjoint union of  $n$  simple knotted contours  $C_j$ ,  $j = 1, 2 \dots n$ . The holonomy operator is the tensor product of those defined on each component  $C_j$

$$f[A, L] \equiv P\exp \left( i \oint_L A \right) = \bigotimes_{j=1}^n P\exp \left( i \oint_{C_j} A \right). \quad (21)$$

Using the same reasoning that was used to derive Eq.(7), we can see that quantum holonomy defined a multi-component link commutes with any appropriately tensored generators of the gauge group,

$$\left[ \bigotimes_{i=1}^n T_{(i)}^a, Z[L] \right] = 0, \quad Z[L] = \langle f(A, L) \rangle = \langle P\exp \left( i \oint_L A \right) \rangle. \quad (22)$$

This shows that  $Z[L]$  is a commutant of the universal enveloping algebra (of the Lie algebra of gauge group). The matrix representation of  $Z[L]$  is not as simple as that of the quantum holonomy of a simple (one-component) knotted contour, however. If we now evaluate  $Z[L]$  in the representation  $\bigotimes_{i=1}^n \rho_i$ , the result will *not* be a polynomial times the  $N$ -dimensional matrix representation of the unity element, where  $N$  is equal to the product of the dimensions of  $\rho_i$ ,  $i = 1, \dots, n$ :

$$N = \prod_{i=1}^n N(\rho_i). \quad (23)$$

This is because  $\bigotimes_{i=1}^n \rho_i$  is reducible. Suppose this tensored representation decomposes as

$$\bigotimes_{i=1}^n \rho_i = \bigoplus_{j=1}^m \tau_j, \quad N = \sum_{j=1}^m N(\tau_j), \quad (24)$$

where each  $\tau_j$  is irreducible, then it follows from Eq.(22) that  $\bigotimes_{i=1}^n \rho_i(Z[L])$  will be a diagonal matrix of dimension  $N$ , with its  $N$  diagonal matrix elements being composed of possibly  $m$  distinct polynomials, each polynomial repeating  $N(\tau_j)$  times.

The above is a general algebraic property of  $Z[L]$ , regardless of whether the theory is topological or not. For a topological field theory, the polynomials in  $\bigotimes_{i=1}^n \rho_i(Z[L])$  carries

additional topological information. In the case of the Chern-Simons theory in three dimensions, the polynomials will carry information pertaining to  $L$  being a member of an isotopy class. Let us take the traces of all the representations in  $\bigotimes_{i=1}^n \rho_i$  except one, say that of  $\rho_k$ . Then we obtain the counterpart of Eq.(7),

$$\bigotimes_{j \neq k} \text{Tr}_{\rho_j} \left( \bigotimes_{i=1}^n \rho_i (Z[L]) \right) = F(L) \mathbf{1}_{\rho_k}, \quad (25)$$

where  $F(L)$  is a polynomial of the  $m$  polynomials in  $\bigotimes_{i=1}^n \rho_i (Z[L])$ ; it is a polynomial for  $L$ . One might think that  $F(L)$  would be labeled by  $\rho_k$ , but it has been shown [19] that Eqs. (22) and (25) are sufficient to prove that  $F(L)$  is independent of the choice of  $\rho_k$ . Thus  $F(L)$  is a uniquely defined eigenvalue of  $Z[L]$  and is a link polynomial for  $L$  on the set of representations  $\{\rho_1, \rho_2, \dots, \rho_n\}$ .

An explicit evaluation of  $Z[L]$  by the right-hand-side of Eq.(22) is nontrivial. This contrasts with the rather straightforward evaluation of the link polynomial  $F(L)$  by algebraic means. This method, based on long standing theorems by Alexander [20] and Reidemeister [21], begins by making a planar projection of a link, called a link diagram, in such a way that the projection is a network whose only nodes are either one of two kinds of crossings - overcrossing or undercrossing - each being a four-valent planar diagram. Topologically equivalent classes of links are classified according to isotopic classes of link diagrams. The connection to a gauge group either through a skein relation [3], the braid group [22] or directly [19] is made by mapping the overcrossing (undercrossing, resp.) to (the inverse of, resp.) an invertible universal  $R$ -matrix, which is a construct (more specifically, a second rank tensor product) that exists in universal enveloping algebras of Lie algebras such as the Lie algebra of  $SU(N)$ .

In Ref. [23], the emergence of an  $R$ -matrix in Wilson loops of the three-dimensional Chern-Simons theory was investigated. It was shown that by choosing a special gauge - the almost axial gauge - and working in the space-time manifold  $S^1 \times R^2$ , one can use the technique of standard perturbation theory to reveal the assignment of an  $R$ -matrix to the crossing on a link diagram. Since the trace of the Wilson loop was actually not taken in Ref.

[23], its conclusions more appropriately apply to quantum holonomy. However, because the object of investigation, the link invariant, is a nonperturbative property of the Chern-Simons theory, the conclusion is somewhat clouded through a lack accuracy owing to the nature of the perturbation method. Perhaps a direct nonperturbative evaluation of Eq.(22), such by lattice gauge theory, is called for.

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